ECON 6110, Microeconomic Theory III	Section 1 Take-home
TA: Ruqing Xu	January 24, 2025

1 Proof of Nash's Theorem (1950)

Before we prove Nash's Theorem, we provide a couple of the needed background definitions. We omit the required definitions of correspondence, closed set, compact set, and convex set; we should all remember these from ECON 6170:

Definition 1: A *fixed point* of the correspondence $\phi : Z \rightrightarrows Z$ is an element $z \in Z$ such that $z \in \phi(z)$.

Definition 2: A correspondence $\phi : W \rightrightarrows Z$ has a *closed graph* if for all sequences of elements $w_n \in W$ and $z_n \in Z$ for which $z_n \in \phi(w_n)$ for all n, $w_n \to w$, and $z_n \to z$, we have that $z \in \phi(w)$.

Lemma (Kakutani's Fixed Point Theorem): For a correspondence $\phi : Z \rightrightarrows Z$, the following conditions are sufficient to guarantee that $\phi(\cdot)$ has a fixed point:

1. Z is a compact, convex, nonempty subset of a finite dimensional Euclidean space.
2. $\phi(z) \neq \emptyset$ for all $z \in Z$.
3. $\phi(z)$ is convex for all $z \in Z$.
4. $\phi(\cdot)$ has a closed graph (or $\phi(\cdot)$ is upper hemicontinuous).

Proof: Google it.[1] ■

Theorem (Nash's Theorem): Every *finite* game has a mixed strategy Nash equilibrium.

In class, Prof. Battaglini proved the theorem in two steps. First, he proved an intermediate theorem that gives another set of conditions on the set of strategies and utility functions

[1] The proof of Kakutani's fixed point theorem is rather involved and will not be covered here. For those of you who are interested in learning the theorem, I recommend an exposition written by Younggeun Yoo of the University of Chicago; you can find it via Google.

that would imply the conditions of the Kakutani's theorem (which would then imply there is a fixed point). Then, he verifies that if we allow for mixed strategies, the conditions of the intermediate theorem indeed hold.

Here we combine the two steps. We will directly verify the conditions of the Kakutani's theorem one by one using notations of a game.

Proof: Consider a finite game with N players and a finite action space $A = \Pi_{i=1}^{N} A_i$. Each player's set of mixed strategies is $\Delta(A_i)$. Define the best response correspondence

$$B_i(\alpha_{-i}) = \text{argmax}_{\alpha_i \in \Delta(A_i)} U_i(\alpha_i, \alpha_{-i}). \tag{*}$$

Notice that B_i is player i's best response correspondence over the set of all profiles of mixed strategies played by the other players. With this, we define

$$B(\alpha) = B_1(\alpha_{-1}) \times ... \times B_n(\alpha_{-n}).$$

Note that

$$B : \Delta(A_1) \times ... \times \Delta(A_n) \rightrightarrows \Delta(A_1) \times ... \times \Delta(A_n)$$

In words, $B(\cdot)$ is a correspondence that maps profiles of mixed strategies to profiles of mixed strategies as best responses. As such, any fixed point of $B(\cdot)$ is a Nash Equilibrium, so if we can prove that all fours conditions of Kakutani's fixed point theorem apply to $B(\cdot)$, we are done. Let us take each of the four conditions in turn.

Condition 1: Let $|A_i| = k$ a finite number. The set $\Delta(A_i)$ is just the probability simplex of dimension $k-1$. We should have learned in ECON 6170 that this set is compact, convex, and nonempty. Since the finite product of compact sets is compact, the finite product of convex sets is convex, and the product of non-empty sets is nonempty, we have that $\Delta(A_1) \times ... \times \Delta(A_n)$ is compact, convex, and nonempty. Furthermore, it resides in a finite dimensional Euclidean space on account of the facts that the probability simplex is a Euclidean subset and our game is finite.

Condition 2: First, observe that for any fixed α_{-i}, the expected utility of player i using mixed strategy α_i is

$$U_i(\alpha_i, \alpha_{-i}) = \sum_{a_i} U_i(a_i, \alpha_{-i}) \alpha_i(a_i)$$

which is the dot product of α_i and another vector, and thus is linear in α_i. Thus, it is continuous over $\Delta(A_i)$. Furthermore, as was mentioned previously, $\Delta(A_i)$ is compact. As such, we can apply the Wierstrass theorem to determine that a solution to the maximization problem (*) exists. Thus, $B_i(\cdot)$ is nonempty over its domain, so $B(\cdot)$ is nonempty over its domain.

Condition 3: For any i and α_{-i}, let β_i and β_i' be any two probability distributions over i's pure strategies such that $\beta_i, \beta_i' \in B_i(\alpha_{-i})$. By definition, β_i and β_i' are both best responses by i to the profile of mixed strategies α_{-i}. Thus, by the indifference condition for mixed strategies, i is indifferent between all pure strategies in the set

$$S = \text{support}(\beta_i) \cup \text{support}(\beta_i'),$$

and thus indifferent to all mixed strategies with support S. Since for any $\theta \in (0,1)$ the mixed strategy $\theta\beta_i + (1-\theta)\beta_i'$ has support S, we have that i is indifferent between $\theta\beta_i + (1-\theta)\beta_i'$ and β_i, so

$$\theta\beta_i + (1-\theta)\beta_i' \in B_i(\alpha_{-i}).$$

This proves that $B_i(\cdot)$ is convex over its domain, so $B(\cdot)$ is convex over its domain.

Condition 4: Define sequences $(\alpha_i^t, \alpha_{-i}^t) \to (\alpha_i, \alpha_{-i})$ with $\alpha_i^t \in B_i(\alpha_{-i}^t)$ for all t. Then, suppose towards a contradiction that $\alpha_i \notin B_i(\alpha_{-i})$. Thus, $\exists \tilde{\alpha}_i$ and $\epsilon > 0$ such that

$$U_i(\tilde{\alpha}_i, \alpha_{-i}) \geq U_i(\alpha_i, \alpha_{-i}) + \epsilon.$$

We next show that $\tilde{\alpha}_i$ is a better response for α_{-i}^t for some t than α_i^t, and thus contradicts $\alpha_i^t \in B_i(\alpha_{-i}^t)$.

For sufficiently large t,

$$\begin{aligned}
U_i(\tilde{\alpha}_i, \alpha_{-i}^t) &\geq U_i(\tilde{\alpha}_i, \alpha_{-i}) - \frac{\epsilon}{2} && (1)\\
&\geq U_i(\alpha_i, \alpha_{-i}) + \epsilon - \frac{\epsilon}{2} && (2)\\
&\geq U_i(\alpha_i^t, \alpha_{-i}^t) - \frac{\epsilon}{4} + \frac{\epsilon}{2} && (3)\\
&= U_i(\alpha_i^t, \alpha_{-i}^t) + \frac{\epsilon}{4} && (4)
\end{aligned}$$

(1) comes from that $\alpha_{-i}^t \to \alpha_{-i}$ and that U_i is continuous. (3) comes from that $(\alpha_i^t, \alpha_{-i}^t) \to (\alpha_i, \alpha_{-i})$ and that U_i is continuous.

The above result contradicts $\alpha_i^t \in B_i(\alpha_{-i}^t)$. Thus, we proved that $B_i(\cdot)$ has a closed graph, so $B(\cdot)$ also has a closed graph.