

1 Extensive game

Definition 1.1. An **extensive game with perfect information** $\Gamma = \langle N, H, P, (u_i) \rangle$ consists of

- (i) (Finite) set of players, $N = \{1, \dots, n\}$
- (ii) A set H of histories with the following properties:
 - (a) An initial history $\emptyset \in H$
 - (b) If $(a^k)_{k=1, \dots, K} \in H$, then $(a^k)_{k=1, \dots, L} \in H$ for all $L < K$.
 - (c) If an infinite sequence $(a^k)_{k=1}^\infty$ satisfies $(a^k)_{k=1}^L \in H$ for every positive integer L , then $(a^k)_{k=1}^\infty \in H$.
- (iii) A function $P : H \setminus Z \rightarrow N$ that assigns to each nonterminal history a member of N .
- (iv) Preferences over terminal histories for all i , $u_i : Z \rightarrow \mathbb{R}$

Definition 1.2. Each member of H is a **history**; each component of a history is an **action** taken by a player. A history $(a^k)_{k=1}^K \in H$ is called **terminal** if it is infinite or if there is no a^{K+1} such that $(a^k)_{k=1}^{K+1} \in H$. The set of terminal histories is denoted Z .

Definition 1.3. A strategy of a player i in an extensive game with perfect information is a function

$$s_i(h) \rightarrow A(h)$$

for any $h \in H \setminus Z$ such that $P(h) = i$.

Remark. A strategy specifies an action for *each* (non-terminal) history in which a player is asked to choose an action, even for histories that, if the strategy is followed, are never reached.

Definition 1.4. Denote a strategy profile $s = (s_1, \dots, s_n)$. For each strategy profile an outcome $O(s)$ is the terminal history associated with the strategy profile.

Definition 1.5. A strategy profile, $s = (s_1, \dots, s_n)$ is a **Nash equilibrium** if for all players i and all deviations \hat{s}_i ,

$$u_i(s_i, s_{-i}) \geq u_i(\hat{s}_i, s_{-i})$$

where $u_i(s) = u_i(O(s))$.

Definition 1.6. The **subgame** of the extensive game with perfect information $\Gamma = \langle N, H, P, (u_i) \rangle$ that follows the history h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (u_i)|_h \rangle$, where $H|_h, P|_h, (u_i)|_h$ are consistent with the original game starting at history h .

Definition 1.7. A strategy profile, s is a **subgame perfect equilibrium** in Γ if for any history h the strategy profile $s|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

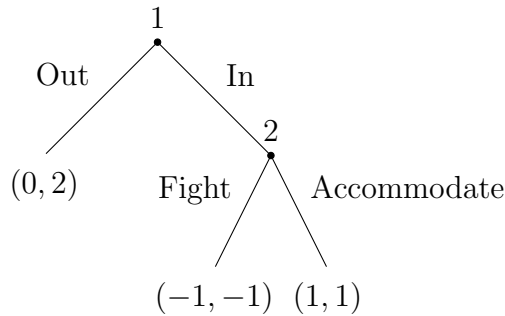
Definition 1.8. For fixed s_i and history h , a **one-stage deviation** is a strategy \hat{s}_i in the subgame $\Gamma(h)$ that differs from $s_i|_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Theorem 1.1 (One-stage deviation principle). *In a finite-horizon extensive game or infinite horizon games continuous at infinity, a strategy profile s is an SPE if and only if for all players i , all histories $h \in H$, and one-stage deviations \hat{s}_i ,*

$$u_i(s_i|_h, s_{-i}|_h) \geq u_i(\hat{s}_i, s_{-i}|_h)$$

Theorem 1.2 (Kuhn's). *SPE for finite extensive games can be found by Backward induction.*

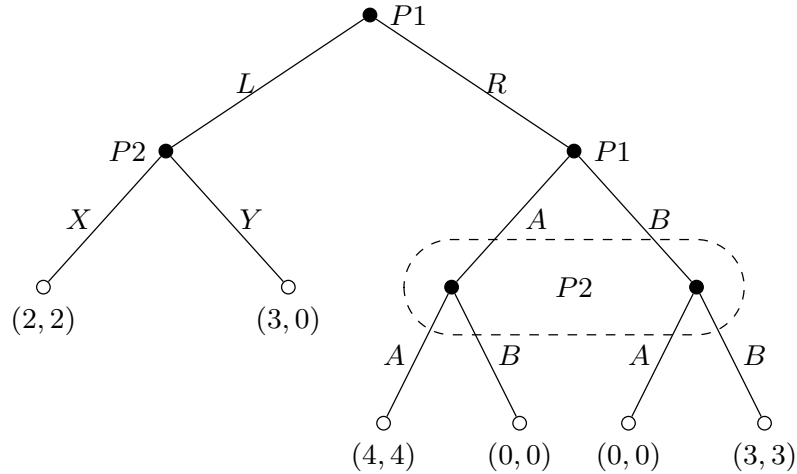
Example 1.3 (Entry game).



2 Microeconomic Theory Qualification Exam, 2018 Retake: Question III

Part III (20 Points)

Consider the following dynamic game in extensive form:



- (3 points) List all pure strategies that each player has.
- (3 points) How many subgames are there? Please describe them.
- (9 points) Find all (pure or mixed) subgame perfect equilibria.
- (5 points) Find a Nash equilibrium that is not subgame perfect.

Proof. (a) A pure strategy specifies what each player does at all of their respective decision nodes. Thus, the pure strategies are

- Player 1: $\{L, R\} \times \{A, B\}$
- Player 2: $\{X, Y\} \times \{A, B\}$.

(b) There are three subgames:

- Subgame 1: The coordination game that occurs after Player 1 plays R
- Subgame 2: Player 2's decision problem that occurs after Player 1 plays L .

- Subgame 3: The entire game.

(c) For a SPNE, a Nash must be played in each proper subgame:

- Subgame 1: The three NE are (A, A) ; (B, B) ; and $(\frac{3}{7}A + \frac{4}{7}B, \frac{3}{7}A + \frac{4}{7}B)$.
- Subgame 2: The Unique NE is (Idle, X)

Thus, if Player 1 plays L , she is guaranteed a payoff of 2, but if she plays R , she can either receive a payoff of 4, 3, or $\frac{12}{7}$ in the case of the interior subgame NE. Thus, the SPNE are as follows:

- $\{(R, A); (X, A)\}$
- $\{(R, B); (X, B)\}$
- $\{(L, \frac{3}{7}A + \frac{4}{7}B); (X, \frac{3}{7}A + \frac{4}{7}B)\}$

(d) The strategy profile

$$\{(R, A); (Y, A)\}$$

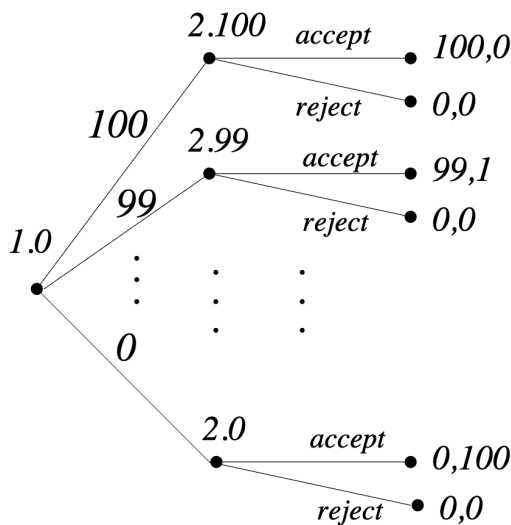
is a NE, but it is not subgame perfect as Player 2 is not playing a best response in Subgame 2.

□

3 Ultimatum Game

Consider the following two stage game, about how two players split a pile of 100 gold coins. The action (strategy) set of player 1 is given by $S_1 = \{0, \dots, 100\}$, with choice i meaning that player 1 proposes to keep i of the gold coins. Player 2 learns the choice of player one, and then takes one of two actions in response: 1 (accept) or 0 (reject). If player two plays accept, the payoff vector is $(i, 100 - i)$. If player two plays reject, the payoff vector is $(0, 0)$.

- Describe the extensive form version of the game using a game tree.
- Describe the normal form of the game. It suffices to specify the strategy spaces and payoff functions. (Hint: Player 2 has 2^{101} pure strategies.)
- Identify a Nash equilibrium of the normal form game with payoff vector $(50, 50)$.
- Identify the subgame perfect equilibria of the extensive form game. (Hint: There are two of them.)
- Do the subgame perfect equilibria change if player 1's strategy space is now continuous, i.e., $S_1 = [0, 100]$?



Proof. (a)

- The strategy spaces are $S_1 = \{0, \dots, 100\}$ and $S_2 = \{0, 1\}^{101}$. A strategy of player 2 can be represented as a vector $d = (d(i) : 0 \leq i \leq 100)$. $d(i) = 1$ if player 2 accepts

when player 1 plays i , and $d(i) = 0$ if player 2 rejects when player 1 plays i .

The payoff functions are given by

$$u_1(i, d) = i \cdot d(i)$$

$$u_2(i, d) = (100 - i) \cdot d(i)$$

- (c) Denote a cutoff strategy of player 2 d_k as $d(i) = 1, \forall i \leq k$ and $d(i) = 0 \forall i > k$. The strategy profile $(50, d_{50})$ is a Nash equilibrium.
- (d) To find SPE we use backward induction. For all subgame $i \leq 99$, the subgame Nash is accept. Only for the last subgame, both accept and reject are Nash. Suppose player 2 always accept, then for player 1 the best response is 100. Suppose player 2 reject only if $i = 100$, then for player 1 the best response is 99. Then, $(99, d_{99})$ and $(100, d_{100})$ are subgame perfect equilibria.
- (e) Now, only $(100, d_{100})$ is subgame perfect equilibrium. If player 2 rejects at $i = 100$, then no strategy of player 1 is Nash.

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